Math 299 **Review for Midterm 1**

PROBLEM 1. Suppose $f : A \to B$ and $g : B \to C$ are functions.

(a) Prove the following statement.

> (*) If f is surjective and g is surjective, then the composition $g \circ f : A \to C$ is surjective.

(b) Identify the hypothesis and conclusion of statement (*), above.

(c) State the inverse, contrapositve and converse of statement (*). Determine whether each of these is true or false. For each true statement, provide a short proof; for each false statement, provide a counterexample.

PROBLEM 2. Prove the statements appearing in (a)-(c), and answer the prompt in (d). The symbol $\cong$ denotes *bijective correspondence.*

(a) For all sets A and B, if $A \cong B$, then $B \cong A$.

(b) Suppose A is a set. Then $A \cong A$.

(c) For all sets A, B and C, if $A \cong B$ and $B \cong C$, then $A \cong C$. *Hint: Use problem 1 above, and one of the homework or essay problems.*

(d) State the negation of each of the statements (a)-(c) above. Determine if the negation is true or false. Provide a counterexample for any false statement.

PROBLEM 3. Let E denote the set of even integers.

(a) Use a picture to illustrate a bijection between $\mathbb{N}$ and $E \times E$.

(b) Use a picture to illustrate a bijection between $\mathbb{Z}$ and $E \times E$.

PROBLEM 4.

(a) Find a set $S \subseteq \mathbb{R}$ such that the function

$$\begin{aligned} f : [0, \infty) &\longrightarrow S \\ x &\longmapsto \frac{1}{1+x^2} \end{aligned}$$

is surjective.

(b) Let f and S be as in part (a). Prove that f is injective.

(c) Let S be as in part (a). Suppose T is a set which is strictly larger than S; that is, $S \subseteq T$, but $S \neq T$. Explain why the function

$$\begin{aligned} g : [0, \infty) &\longrightarrow T \\ x &\longmapsto \frac{1}{1+x^2} \end{aligned}$$

is not surjective.

(d) Let S be as in part (a). Suppose R is a set which is strictly smaller than S; that is, $R \subseteq S$, but $R \neq S$. Explain why there is no function of the form

$$\begin{aligned} e : [0, \infty) &\longrightarrow R \\ x &\longmapsto \frac{1}{1+x^2} \end{aligned}$$

PROBLEM 5. An expression involving quantifieres is in *positive form* if none of the quantifiers is negated. Thus $\neg\forall x,\ P(x)$ is not in positive form, but the equivalent expression $\exists x,\ \neg P(x)$ is in positive form. Negate each of the following statements and express it in positive form.

1. $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N},\ x + y = 1.$

2. $\forall x > 0\ \exists y < 0,\ x + y = 0.$

3. $\exists x \in \mathbb{R}\ \forall \epsilon > 0,\ -\epsilon < x < \epsilon.$

4. $\forall x, y \in \mathbb{N}\ \exists z \in \mathbb{N},\ x + y = z^2.$

PROBLEM 6. Negate the following.

(a) $\forall n \in \mathbb{Z}, \quad \exists m \in \mathbb{Z}$ such that $m \cdot n = 1.$

(b) $\exists x \in \mathbb{Q}$ such that $\forall y \in Q, \quad x \cdot y = y.$

Rewrite the statements in (a) and (b) without the use of quantifiers and state if it is a true or a false statement. If it is a true statement, prove it. If it is a false statement, provide a counterexample.

PROBLEM 7. Construct a truth table to show that the contrapositive of $A \Rightarrow B$ is equivalent to $A \Rightarrow B$.

PROBLEM 8. Prove the following statement.

$$\forall a \in \mathbb{R}\, \exists! x \in \mathbb{R}, \text{ such that } 3x - 1 = a.$$

PROBLEM 9 Fill in the blank with *necessary, sufficient*, or *necessary and sufficient.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

$(f'(a) = 0)$ is a ____ condition for (f has a local maximum or local minimum at $x = a$).

(a) Write it as an "if ... then ..." or "... if and only if ..." statement.

(b) Write the converse of the statement in (2).

(c) Write the contrapositive of the statement in (2).

(d) Write the inverse of the statement in (2).

(e) If the implication in (3) is False, then illustrate why it is False with an example.

PROBLEM 10. Let E denote the set of even integers and A be the following statement.

$$A: \text{“}x \in E \Rightarrow \exists k \in \mathbb{Z} \text{ such that } x = 2k\text{”}$$

(a) Write the inverse of statement A.

(b) Write the converse of statement A.

(c) Write the contrapositive of statement A.

(d) Is statement A true? What about its converse? In this case, how would you restate it using *necessary/ sufficient/ necessary and sufficient*?

(e) Which of the statements in parts (a), (b), or (c) is equivalent to the original statement in general (no matter what A is)?

PROBLEM 11. Let $A = \{x \in \mathbb{Z} | x = 6k, k \in \mathbb{Z}\}$, $B = \{x \in \mathbb{Z} | x = 2k, k \in \mathbb{Z}\}$, $C = \{x \in \mathbb{Z} | x = 3k, k \in \mathbb{Z}\}$. Prove the following statement.

$$x \in A \iff \exists y \in B \text{ and } \exists z \in C \text{ such that } x = yz.$$